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Asymptotic behavior of the joint record values, with applications

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ABSTRACT

The class of limit distribution functions of the joint upper record values, as well as the joint of lower record values, is fully characterized. Sufficient conditions for the weak convergence are obtained. As an application of this result, the sufficient conditions for the weak convergence of the record quasi-range, record quasi-midrange, record extremal quasi-quotient and record extremal quasi-product are obtained. Moreover, the classes of the non-degenerate limit distribution functions of these statistics are derived.

1. Introduction

Record values arise naturally in many practical problems and there are several situations pertaining to meteorology, hydrology, sporting and athletic events wherein only record values may be recorded. Suppose that $\{X_n,\ n \geq 1\}$ is a sequence of mutually independent random variables (rv's) with common distribution function (df) $F(x)$. We say X_j is an upper record value of $\{X_n, n \geq 1\}$, if $X_{j:j} > X_{j-1:j-1},\ j > 1$. An analogous definition deals with lower record values. By definition X_1 is an upper as well as lower record value. Thus, the upper and the lower record values in the sequence $\{X_n,\ n \geq 1\}$ are the successive maxima and the successive minima, respectively. The upper (lower) record time sequence $\{N_n,\ n \geq 1\}$ ($\{M_n,\ n \geq 1\}$) is defined by $N_n = \min\{j : j > N_{n-1}, X_j > X_{N_{n-1}}, n > 1\}$ and $N_1 = 1$ ($M_n = \min\{j : j > M_{n-1}, X_j < X_{M_{n-1}}, n > 1\}$ and $M_1 = 1$). Then the upper (lower) record value sequence $\{R_n\}$($\{L_n\}$) is defined by $R_n = X_{N_n}$ ($L_n = X_{M_n}$) and it can be expressed in terms of the function $h(x) = -\log \overline{F}(x)$ $\left(\tilde{h}(x) = -\log F(x)\right)$, where $\overline{F}(x) = 1 - F(x)$, e.g., the exact df of the upper (lower) record value is given by (cf. Arnold et al., 1998)

$$P(R_n \leq x) = \Gamma_n(h(x)) \left(P(L_n \leq x) = \Gamma_n(\tilde{h}(x))\right), \quad n > 1,$$

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where $\Gamma_n(x) = \frac{1}{\Gamma(n)}\int_0^x t^{n-1}e^{-t}dt$ is the incomplete gamma ratio function. The well-known asymptotic relation $\Gamma_n(\sqrt{n}x + n)\underset{n}{\rightarrow}\mathcal{N}(x)$ ($\underset{n}{\rightarrow}$ stands for convergence, as $n \to \infty$), for all real values of x, where $\mathcal{N}(x)$ is the standard normal distribution, enables us to deduce the following basic result, which is originally due to Tata (1969) (see also Corollary 6.4.1 in Galambos, 1987).

Lemma 1.1. *Let s_n be a sequence of integer numbers, such that $s_n\underset{n}{\rightarrow}\infty$. Then, there are constants $a_n, \tilde{a}_n > 0$ and $b_n, \tilde{b}_n \in \mathbb{R}$, such that*

$$\Phi_{R_{s_n}}(a_n x + b_n) = P(R_{s_n} \le a_n x + b_n)\overset{w}{\underset{n}{\longrightarrow}}\Phi_R(x) \tag{1.1}$$

and

$$\Phi_{L_{s_n}}(\tilde{a}_n x + \tilde{b}_n) = P(L_{s_n} \le \tilde{a}_n x + \tilde{b}_n)\overset{w}{\underset{n}{\longrightarrow}}\Phi_L(x) \tag{1.2}$$

($\overset{w}{\underset{n}{\longrightarrow}}$ stands for weak convergence, as $n \to \infty$), where $\Phi_R(x)$ and $\Phi_L(x)$ are non-degenerate df's, if and only if $\frac{h(a_n x+b_n)-s_n}{\sqrt{s_n}}\underset{n}{\rightarrow}V(x)$ and $\frac{\tilde{h}(\tilde{a}_n x+\tilde{b}_n)-s_n}{\sqrt{s_n}}\underset{n}{\rightarrow}V(-x)$, respectively, where $V(x)$ is finite on an interval and has at least two growth points. In this case we have $\Phi_R(x) = \mathcal{N}(V(x))$ and $\Phi_L(x) = 1 - \mathcal{N}(V(-x))$.

Resnick (1973) showed that the function $V(.)$ can only take three possible types (denoted by $V_j(x;\gamma)$, $j = 1, 2, 3$, $\gamma > 0$), or equivalently there are only three kinds of distributions that could arise as limiting distributions of suitably normalized upper (lower) record values. Namely, the only possible limiting distributions of suitably normalized upper record value are $H_{j,\gamma}(x) = \mathcal{N}(V_j(x;\gamma))$, $\gamma > 0, j = 1, 2$, and $H_{3,\gamma}(x) = H_{3,0}(x) = \mathcal{N}(V_3(x))$, where

$$V_1(x;\gamma) = \begin{cases} -\infty, & x < 0, \\ \gamma \log x, & x \ge 0; \end{cases} \qquad V_2(x;\gamma) = \begin{cases} -\gamma \log |x|, & x < 0, \\ \infty, & x \ge 0; \end{cases}$$
$$V_3(x;\gamma) = V_3(x) = x, \quad \forall x. \tag{1.3}$$

In this case we say that F is in the domain of upper record attraction of $H_{j,\gamma}$ and write $F \in \mathcal{D}_{\mathcal{R}}(H_{j,\gamma})$. Throughout this paper, we will assume that $F \in \mathcal{D}_{\mathcal{R}}(H_{j,\gamma}), j \in \{1, 2, 3\}$. The following theorem due to Resnick (1973) (see also Arnold et al., 1998) is needed in our study:

Theorem 1.1 (*The Duality Theorem*). *If F is a continuous df with an associated df F_a, which is defined by $F_a(x) = 1 - \exp(-\sqrt{h(x)})$, and $\Psi_F(n) = inf\{y : F(y) > 1 - e^{-n}\} = F^{-1}(1 - e^{-n})\underset{n}{\rightarrow}x^0 = \sup\{x : F(x) < 1\}$, then the following limit implications hold:*

(i) *$F \in \mathcal{D}_{\mathcal{R}}(H_{1,\gamma})$ if and only if $F_a \in \mathcal{D}(G_{1,\frac{\gamma}{2}})$. In this case $F^{-1}(1) = x^0 = \infty$ and we may use as normalizing constants $a_n = \Psi_F(n)$ and $b_n = 0$;*
(ii) *$F \in \mathcal{D}_{\mathcal{R}}(H_{2,\gamma})$ if and only if $F_a \in \mathcal{D}(G_{2,\frac{\gamma}{2}})$. In this case $F^{-1}(1) = x^0$ is necessarily finite and we may use as normalizing constants $a_n = x^0 - \Psi_F(n)$ and $b_n = x^0$;*
(iii) *$F \in \mathcal{D}_{\mathcal{R}}(H_{3,0})$ if and only if $F_a \in \mathcal{D}(G_{3,0})$ and in this case we may use as normalizing constants $a_n = \Psi_F(n + \sqrt{n}) - \Psi_F(n)$ and $b_n = \Psi_F(n)$,*

where $G_{j,\gamma}(x) = \exp(-\exp(-(V_j(x;\gamma))))$, $j = 1, 2, 3$, are the well-known limit distributions of the maximum order statistics (see, Galambos, 1987).

Our aim in this paper is to study the asymptotic behavior of the joint upper (lower) record values. This problem is recently tackled by Barakat et al. (2014), for m-generalized order statistics when $m > -1$, i.e., the record values case was excluded from this study. In this paper we will fill this gap. Moreover, as application of this study, we could study the asymptotic behavior of some simple functions of record values, which have important applications. Namely, the record quasi-range $W_n = R_{s_n} - R_{r_n}$, the record quasi-midrange $M_n = \frac{R_{s_n}+R_{r_n}}{2}$, the record extremal quasi-quotient $Q_n = \frac{R_{s_n}}{R_{r_n}}$ and the record extremal quasi-product $P_n = R_{s_n}R_{r_n}$, where $r_n < s_n$ are two sequences of integers. We derive the possible non-trivial and trivial limit df's of all suitably normalized preceding statistics, the trivial limit is defined when the convergence takes place, such that one of the statistics R_{s_n} and R_{r_n} outweighs the other (see De Haan, 1974). This problem is recently tackled by Barakat et al. (2015a), for m-generalized order statistics when $m > -1$, i.e., the record values case was excluded from this study. Moreover, the same problem is studied for record values in Barakat et al. (2015b), but for the following special cases: the record range $W_n = R_n - R_1 = R_n - X_1$, the record midrange $M_n = \frac{R_n+R_1}{2} = \frac{R_n+X_1}{2}$, the record extremal quotient $Q_n = \frac{R_n}{R_1} = \frac{R_n}{X_1}$ and the record extremal product $P_n = R_nR_1 = R_nX_1$. El Arrouchi (2016) is a recent relevant work to the asymptotic behavior of functions of record values, where some specific characterization results of tail df's by the ratios of the successive record values are obtained. Moreover, Gut and Stadtmüller (2016) studied the weak convergence of counting variables for record values and the record times.

2. Asymptotic theory of joint record values

In this section we study the asymptotic behavior of the joint df of R_{s_n} and R_{r_n}, as well as the joint df of L_{s_n} and L_{r_n}, where $r_n < s_n$ are two sequences of integers. It is well-known that the asymptotic independence between any two upper records R_{r_n} and R_{s_n}, as well as any two lower records L_{s_n} and L_{r_n}, occurs if and only if $s_n - r_n \underset{n}{\to} \infty$ (cf. Barakat, 2007). Therefore, the only correct way to study the asymptotic behavior of the joint df $\Phi_{R_{r_n},R_{s_n}}(x_n, y_n) = P(R_{r_n} \le x_n, R_{s_n} \le y_n) = P(R_{r_n} \le \frac{x-b_n}{a_n}, R_{s_n} \le \frac{y-d_n}{c_n})$ ($\Phi_{L_{r_n},L_{s_n}}(\tilde{x}_n, \tilde{y}_n) = P(L_{r_n} \le \tilde{x}_n, L_{s_n} \le \tilde{y}_n) = P(L_{r_n} \le \frac{x-\tilde{b}_n}{\tilde{a}_n}, L_{s_n} \le \frac{y-\tilde{d}_n}{\tilde{c}_n})$), where $a_n, c_n > 0$ and $b_n, d_n \in \mathbb{R}$ ($\tilde{a}_n, \tilde{c}_n > 0$ and $\tilde{b}_n, \tilde{d}_n \in \mathbb{R}$) are suitable normalizing constants, is to consider the following three disjoint and exhausted cases:

1. $r = r_n$ is constant with respect to n and $s_n \underset{n}{\to} \infty$.
2. Both of the indices tend to infinity (i.e., $s_n, r_n \underset{n}{\to} \infty$) and $s_n - r_n \underset{n}{\to} \infty$.
3. Both of the indices tend to infinity and $s_n - r_n \underset{n}{\to} m$, where m is a finite integer.

Theorem 2.1 (*The Asymptotic Behavior of the Joint Upper Record Values*). *Let $a_n > 0$ and b_n be normalizing constants for which (1.1) is satisfied. Furthermore, let $\{r_n\}$ and $\{s_n\}$ be subsequences of $\{n\}$, such that $r_n < s_n$. Finally, let $x_n = a_{r_n}x + b_{r_n}$ and $y_n = a_{s_n}y + b_{s_n}$. Then,*

1. *$\Phi_{R_{r_n},R_{s_n}}(x, y_n) \overset{w}{\underset{n}{\to}} \Gamma_r(h(x))\mathcal{N}(V_j(y;\gamma)), j \in \{1,2,3\}$, if $r = r_n$ is constant with respect to n and $s_n \underset{n}{\to} \infty$.*
2. *Moreover, $\Phi_{R_{r_n},R_{s_n}}(x_n, y_n) \overset{w}{\underset{n}{\to}} \mathcal{N}(V_j(x;\gamma))\mathcal{N}(V_j(y;\gamma)), j \in \{1,2,3\}$, if $s_n, r_n \underset{n}{\to} \infty$ and $s_n - r_n \underset{n}{\to} \infty$.*
3. *Finally, $\Phi_{R_{r_n},R_{s_n}}(x_n, y_n) \overset{w}{\underset{n}{\to}} \begin{cases} \mathcal{N}(V_j(y;\gamma)), & y \le x, \\ \mathcal{N}(V_j(x;\gamma)), & x < y, j \in \{1,2,3\}, \end{cases}$ if both of the indices tend to infinity and $s_n - r_n \underset{n}{\to} m$, where $m \ge 1$.*

Proof. The proof of Part (2) follows immediately by applying Lemma 1.1 and by using the fact that the asymptotic independence between the upper records R_{r_n} and R_{s_n} occurs if and only if $s_n - r_n \underset{n}{\to} \infty$. Moreover, by the same argument we get $\Phi_{R_r,R_{s_n}}(x, y_n) \overset{w}{\underset{n}{\to}} \Gamma_r(h(x))\mathcal{N}(V_j(y;\gamma))$, which implies the result of Part (1). To prove Part (3), we first consider the fact (cf. Arnold et al., 1998) that the record values $R^\star_{s_n}$ and $R^\star_{r_n}$ from exponential distribution can be seen as sums of i.i.d. exponential rv's, i.e., as $R^\star_{s_n} = \sum_{i=1}^{s_n} Z_i$ and $R^\star_{r_n} = \sum_{i=1}^{r_n} Z_i$. Thus, $R^\star_{s_n}$ and $R^\star_{r_n}$ are gamma distributed. Let $\phi_{R^\star_{r_n}}(.)$ be the probability density function (pdf) of $R^\star_{r_n}$. Then, by using the continuous total probability rule, we get

$$\begin{aligned}\Phi_{R_{r_n},R_{s_n}}(x_n, y_n) &= P(R^\star_{r_n} \le -\log \overline{F}(x_n), R^\star_{s_n} \le -\log \overline{F}(y_n)) \\ &= \int_0^{h(x_n)} P(R^\star_{s_n-r_n} \le h(y_n) - w)\phi_{R^\star_{r_n}}(w)dw \\ &= \frac{1}{(r_n-1)!}\int_0^{h(x_n)} \Gamma_{s_n-r_n}(h(y_n)-w)\, w^{r_n-1}e^{-w}dw.\end{aligned}$$

Thus, the joint df of the normalized statistics R_{r_n} and R_{s_n}, is given by

$$\Phi_{R_{r_n},R_{s_n}}(x_n, y_n) = \begin{cases} \Gamma_{s_n}(h(y_n)), & y_n \le x_n, \\ \dfrac{1}{(r_n-1)!}\displaystyle\int_0^{h(x_n)} \Gamma_{s_n-r_n}(h(y_n)-w)\, w^{r_n-1}e^{-w}dw, & x_n < y_n. \end{cases} \tag{2.1}$$

Now, for large n, we can show that the two inequalities $x < y$ and $x \ge y$ imply the two inequalities $x_n < y_n$ and $x_n \ge y_n$, respectively. Indeed, since we have $s_n - r_n \underset{n}{\to} m$, then, for all x, y, for which $V_j(x;\gamma)$ and $V_j(y;\gamma)$ are finite, we get

$$V_j(y;\gamma) - V_j(x;\gamma) = \lim_{n\to\infty} \frac{h(y_n) - h(x_n) - (s_n - r_n)}{\sqrt{s_n}} = \lim_{n\to\infty} \frac{h(y_n) - h(x_n)}{\sqrt{s_n}},$$

which implies that $h(y_n) - h(x_n) \underset{n}{\to} +\infty$, if $x < y$ and $h(y_n) - h(x_n) \underset{n}{\to} -\infty$, if $y \le x$. This equivalent to the two inequalities $x < y$ and $x \ge y$ imply $x_n < y_n$ and $x_n \ge y_n$, respectively, for large n (since the function $V_j(.)$ is monotone increasing and the function $h(.)$ is monotone non-decreasing). By using this fact and the relation (2.1), we immediately get the proof of Part (3), when $y \le x$. On the other hand, for all $x_n < y_n$, Eq. (2.1) clearly yields the following inequalities

$$\Gamma_{s_n-r_n}(h(y_n) - h(x_n))\Gamma_{r_n}(h(x_n)) \le \Phi_{R_{r_n},R_{s_n}}(x_n, y_n) \le \Gamma_{s_n-r_n}(h(y_n))\Gamma_{r_n}(h(x_n)). \tag{2.2}$$

Since, $h(y_n) \underset{n}{\to} \infty$ and $s_n - r_n \underset{n}{\to} m$, then the right hand side of the inequality (2.2) weakly converges to $\mathcal{N}(V_j(x;\gamma))$. Moreover, since $h(y_n) - h(x_n) \underset{n}{\to} +\infty$, if $x < y$, then the left hand side of the inequality (2.2) also weakly converges to $\mathcal{N}(V_j(x;\gamma))$. The proof is completed. □

Theorem 2.2 (*The Asymptotic Behavior of the Joint Lower Record Values*). *Let $\tilde{a}_n > 0$ and $\tilde{b}_n$ be normalizing constants for which (1.2) is satisfied. Furthermore, let $\{r_n\}$ and $\{s_n\}$ be subsequences of $\{n\}$, such that $r_n < s_n$. Finally, let $\tilde{x}_n = \tilde{a}_{r_n}x + \tilde{b}_{r_n}$ and $\tilde{y}_n = \tilde{a}_{s_n}y + \tilde{b}_{s_n}$. Then,*

1. $\Phi_{L_{r_n},L_{s_n}}(x,\tilde{y}_n)\xrightarrow[n]{w}\Gamma_r(\tilde{h}(x))\,(1-\mathcal{N}(V_j(-y;\gamma))), j\in\{1,2,3\}$, *if* $r=r_n$ *is constant with respect to* n *and* $s_n\xrightarrow[n]{}\infty$.
2. *Moreover,* $\Phi_{L_{r_n},L_{s_n}}(\tilde{x}_n,\tilde{y}_n)\xrightarrow[n]{w}(1-\mathcal{N}(V_j(-x;\gamma)))\,(1-\mathcal{N}(V_j(-y;\gamma))), j\in\{1,2,3\}$, *if* $s_n, r_n\xrightarrow[n]{}\infty$ *and* $s_n-r_n\xrightarrow[n]{}\infty$.
3. *Finally,* $\Phi_{L_{r_n},L_{s_n}}(\tilde{x}_n,\tilde{y}_n)\xrightarrow[n]{w}\begin{cases}1-\mathcal{N}(V_j(-x;\gamma)), & x\le y,\\ 1-\mathcal{N}(V_j(-y;\gamma)), & y<x, j\in\{1,2,3\},\end{cases}$
   if both of the indices tend to infinity and $s_n-r_n\xrightarrow[n]{}m$, *where* $m\ge 1$.

Proof. The proof of Part (2) follows immediately by applying Lemma 1.1 and by using the fact that the asymptotic independence between the lower records L_{s_n} and L_{r_n} occurs if and only if $s_n-r_n\xrightarrow[n]{}\infty$. Moreover, by the same argument we get $\Phi_{L_r,L_{s_n}}(x,\tilde{y}_n)\xrightarrow[n]{w}\Gamma_r(\tilde{h}(x))(1-\mathcal{N}(V_j(-y;\gamma)))$, which implies the result of Part (1). To prove Part (3), we first consider the joint pdf of the normalized L_{r_n} and L_{s_n}, for $r_n<s_n$, which is given by (cf. Ahsanullah, 1995)

$$\phi_{L_{r_n},L_{s_n}}(\tilde{x}_n,\tilde{y}_n)=\frac{[\tilde{h}(\tilde{x}_n)]^{r_n-1}}{(r_n-1)!}\frac{[\tilde{h}(\tilde{y}_n)-\tilde{h}(\tilde{x}_n)]^{s_n-r_n-1}}{(s_n-r_n-1)!}\frac{f(\tilde{x}_n)f(\tilde{y}_n)}{F(\tilde{x}_n)},\quad \tilde{y}_n<\tilde{x}_n.$$

Therefore, the joint df of the normalized lower record values L_{r_n} and L_{s_n}, for $\tilde{y}_n<\tilde{x}_n$, is given by

$$\begin{aligned}\Phi_{L_{r_n},L_{s_n}}(\tilde{x}_n,\tilde{y}_n)&=P(L_{r_n}\le\tilde{x}_n,L_{s_n}\le\tilde{y}_n)\\&=\int_{-\infty}^{\tilde{y}_n}\int_{v}^{\tilde{x}_n}\frac{[-\log F(u)]^{r_n-1}}{(r_n-1)!}\frac{[-\log F(v)+\log F(u)]^{s_n-r_n-1}}{(s_n-r_n-1)!}(F(u))^{-1}f(u)f(v)dudv.\end{aligned}$$

Let $U=F(u)$ and $V=F(v)$, we obtain

$$\Phi_{L_{r_n},L_{s_n}}(\tilde{x}_n,\tilde{y}_n)=\int_0^{F(\tilde{y}_n)}\int_V^{F(\tilde{x}_n)}\frac{[-\log U]^{r_n-1}}{(r_n-1)!}\frac{[-\log V+\log U]^{s_n-r_n-1}}{(s_n-r_n-1)!}U^{-1}dUdV.$$

By using the transformation $w=-\log U$, $z=-\log V$ (and then use the transformation $t=\frac{w}{z}$), we get

$$\begin{aligned}\Phi_{L_{r_n},L_{s_n}}(\tilde{x}_n,\tilde{y}_n)&=\int_{-\log F(\tilde{y}_n)}^{\infty}\int_{-\log(F(\tilde{x}_n))}^{z}\frac{w^{r_n-1}}{(r_n-1)!}\frac{(z-w)^{s_n-r_n-1}}{(s_n-r_n-1)!}e^{-z}dwdz\\&=\frac{1}{(r_n-1)!(s_n-r_n-1)!}\int_{-\log F(\tilde{y}_n)}^{\infty}z^{s_n-1}e^{-z}\left[\int_{\frac{-\log F(\tilde{x}_n)}{z}}^{1}t^{r_n-1}(1-t)^{s_n-r_n-1}dt\right]dz\\&=\frac{1}{(s_n-1)!}\int_{-\log F(\tilde{y}_n)}^{\infty}z^{s_n-1}e^{-z}\left[1-I_{\frac{-\log F(\tilde{x}_n)}{z}}(r_n,s_n-r_n)\right]dz\\&=1-\Gamma_{s_n}(\tilde{h}(\tilde{y}_n))-\frac{1}{(s_n-1)!}\int_{\tilde{h}(\tilde{y}_n)}^{\infty}z^{s_n-1}e^{-z}I_{\frac{\tilde{h}(\tilde{x}_n)}{z}}(r_n,s_n-r_n)dz,\end{aligned}$$

where $I_x(a,b)=\frac{1}{\beta(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta ratio function. Thus, the joint df of the normalized statistics L_{r_n} and L_{s_n}, is given by

$$\Phi_{L_{r_n},L_{s_n}}(\tilde{x}_n,\tilde{y}_n)=\begin{cases}\Gamma_{r_n}(\tilde{h}(\tilde{x}_n)), & \tilde{x}_n\le\tilde{y}_n,\\ 1-\Gamma_{s_n}(\tilde{h}(\tilde{y}_n))-\dfrac{1}{(s_n-1)!}\displaystyle\int_{\tilde{h}(\tilde{y}_n)}^{\infty}z^{s_n-1}e^{-z}I_{\frac{\tilde{h}(\tilde{x}_n)}{z}}(r_n,s_n-r_n)dz, & \tilde{y}_n<\tilde{x}_n.\end{cases}\tag{2.3}$$

Now, for large n, we can show that the two inequalities $x\le y$ and $x>y$ imply the two inequalities $\tilde{x}_n\le\tilde{y}_n$ and $\tilde{x}_n>\tilde{y}_n$, respectively. Indeed, for all x,y, for which $\tilde{V}_j(x;\gamma)=V_j(-x;\gamma)$ and $\tilde{V}_j(y;\gamma)=V_j(-y;\gamma)$ are finite, we get

$$\tilde{V}_j(y;\gamma)-\tilde{V}_j(x;\gamma)=\lim_{n\to\infty}\frac{\tilde{h}(\tilde{y}_n)-\tilde{h}(\tilde{x}_n)-(s_n-r_n)}{\sqrt{s_n}}=\lim_{n\to\infty}\frac{\tilde{h}(\tilde{y}_n)-\tilde{h}(\tilde{x}_n)}{\sqrt{s_n}},$$

which implies that $\tilde{h}(\tilde{y}_n)-\tilde{h}(\tilde{x}_n)\xrightarrow[n]{}+\infty$, if $y<x$ and $\tilde{h}(\tilde{y}_n)-\tilde{h}(\tilde{x}_n)\xrightarrow[n]{}-\infty$, if $x\le y$. This equivalent to the two inequalities $x\le y$ and $x>y$ implies $\tilde{x}_n\le\tilde{y}_n$ and $\tilde{x}_n>\tilde{y}_n$, respectively, for large n (since the function $\tilde{V}_j(.)$ is monotone decreasing and the function $\tilde{h}(.)$ is monotone non-increasing). By using this fact and the relation (2.3), we immediately get the proof of Part (3), when $x\le y$. On the other hand, for all $\tilde{y}_n<\tilde{x}_n$, Eq. (2.3) clearly yields the following inequalities

$$\begin{aligned}1-\Gamma_{s_n}(\tilde{h}(\tilde{y}_n))\left(1-I_{\frac{\tilde{h}(\tilde{x}_n)}{\tilde{h}(\tilde{y}_n)}}(1,s_n-r_n)\right)&\le 1-\Gamma_{s_n}(\tilde{h}(\tilde{y}_n))-\frac{1}{(s_n-1)!}\int_{\tilde{h}(\tilde{y}_n)}^{\infty}z^{s_n-1}e^{-z}I_{\frac{\tilde{h}(\tilde{x}_n)}{z}}(1,s_n-r_n)dz\\&\le 1-\Gamma_{s_n}(\tilde{h}(\tilde{y}_n))-\frac{1}{(s_n-1)!}\int_{\tilde{h}(\tilde{y}_n)}^{\infty}z^{s_n-1}e^{-z}I_{\frac{\tilde{h}(\tilde{x}_n)}{z}}(r_n,s_n-r_n)dz\\&\le\Phi_{L_{r_n},L_{s_n}}(\tilde{x}_n,\tilde{y}_n)\le 1-\Gamma_{s_n}(\tilde{h}(\tilde{y}_n)).\end{aligned}\tag{2.4}$$

Clearly, the right hand side of the inequality (2.4) weakly converges to $1-\mathcal{N}(V_j(-y;\gamma))$. On the other hand, since, $s_n-r_n\underset{n}{\rightarrow}m$ and under the conditions of Theorem 2.2, $\alpha(F)\underset{n}{\leftarrow}\tilde{y}_n<\tilde{x}_n\underset{n}{\rightarrow}\alpha(F)$, where $\alpha(F)=\inf\{x:F(x)\geq 0\}\geq-\infty$ is the left end-point of the df F, then $\frac{\tilde{h}(\tilde{x}_n)}{\tilde{h}(\tilde{y}_n)}\underset{n}{\rightarrow}0$, for all $y<x$. Thus, the left hand side of the inequality (2.4) also weakly converges to the limit $1-\mathcal{N}(V_j(-y;\gamma))$. This completes the proof of Theorem 2.2. □

3. Application: weak convergence of some record functions

Let $A_{n:t}>0$ and $B_{n:t}\in\mathbb{R}$, $t=w,m,q,p$, be suitable normalizing constants. Furthermore, let $W_n^*=A_{n:w}^{-1}(W_n-B_{n:w})$, $M_n^*=A_{n:m}^{-1}(M_n-B_{n:m})$, $Q_n^*=A_{n:q}^{-1}(Q_n-B_{n:q})$ and $P_n^*=A_{n:p}^{-1}(P_n-B_{n:p})$. The following two theorems fully characterize the possible limit non-degenerate df's (trivial and non-trivial) of the statistics W_n^*, M_n^*, Q_n^* and P_n^*, in the case $s_n-r_n\underset{n}{\rightarrow}\infty$.

Theorem 3.1. *Let $r_n=r=$ constant*

1. *If $F\in\mathcal{D}_{\mathcal{R}}(H_{1,\gamma})$, then $P(W_n^*\leq w)\overset{w}{\underset{n}{\rightarrow}}H_{1,\gamma}(w)$ and $P(M_n^*\leq m)\overset{w}{\underset{n}{\rightarrow}}H_{1,\gamma}(m)$, where the two limit laws are trivial, since R_{s_n} outweighed R_r. In this case, the normalizing constants can be chosen such as $2A_{n:m}=A_{n:w}=a_{s_n}=\Psi_F(s_n)$ and $B_{n:m}=B_{n:w}=0$. Moreover,*

$$P(Q_n^*\leq q)\overset{w}{\underset{n}{\rightarrow}}\begin{cases}\Gamma_r(h(0))+\int_0^\infty H_{1,\gamma}(qt)d\Gamma_r(h(t)), & \text{if } q\geq 0,\\ \int_{-\infty}^0 \overline{H}_{1,\gamma}(qt)d\Gamma_r(h(t)), & \text{if } q<0\end{cases}$$

and

$$P(P_n^*\leq p)\overset{w}{\underset{n}{\rightarrow}}\begin{cases}\Gamma_r(h(0))+\int_0^\infty H_{1,\gamma}\left(\frac{p}{t}\right)d\Gamma_r(h(t)), & \text{if } p\geq 0,\\ \int_{-\infty}^0 \overline{H}_{1,\gamma}\left(\frac{p}{t}\right)d\Gamma_r(h(t)), & \text{if } p<0,\end{cases}$$

where $\overline{H}_{1,\gamma}(.)=1-H_{1,\gamma}(.)$ and we can take $A_{n:q}=A_{n:p}=a_{s_n}=\Psi_F(s_n)$ and $B_{n:q}=B_{n:p}=0$.

2. *If (a) $F\in\mathcal{D}_{\mathcal{R}}(H_{2,\gamma})$, $x^0>0$, or (b) $F\in\mathcal{D}_{\mathcal{R}}(H_{3,0})$, $0<x^0<\infty$, then*

$$P(W_n^*\leq w)\overset{w}{\underset{n}{\rightarrow}}\overline{\Gamma}_r(h(-x^0w)),\quad w\geq 0,\ (\text{trivial limit law, since } R_r \text{ outweighed } R_{s_n}),$$

$$P(M_n^*\leq m)\overset{w}{\underset{n}{\rightarrow}}\Gamma_r(h(x^0m))\ (\text{trivial limit law, since } R_r \text{ outweighed } R_{s_n}),$$

$$P(Q_n^*\leq q)\overset{w}{\underset{n}{\rightarrow}}P\left(\frac{1}{R_r}\leq q+1\right)\ (\text{trivial limit law, since } R_r \text{ outweighed } R_{s_n}),$$

and

$$P(P_n^*\leq p)\overset{w}{\underset{n}{\rightarrow}}P(R_r\leq p+1)=\Gamma_r(h(p+1))\ (\text{trivial limit law, since } R_r \text{ outweighed } R_{s_n}),$$

where $\overline{\Gamma}_r(.)=1-\Gamma_r(.)$ and the normalizing constants can be chosen such as $2A_{n:m}=A_{n:w}=A_{n:q}=A_{n:p}=b_{s_n}$ and $B_{n:w}=B_{n:m}=B_{n:q}=B_{n:p}=b_{s_n}$.

3. *If $F\in\mathcal{D}_{\mathcal{R}}(H_{3,0})$, $x^0=\infty$ and $a_{s_n}^{-1}=(\Psi_F(s_n+\sqrt{s_n})-\Psi_F(s_n))^{-1}\underset{n}{\rightarrow}K<\infty$, then*

$$P(W_n^*\leq w)\overset{w}{\underset{n}{\rightarrow}}\begin{cases}H_{3,0}(w), & \text{if } K=0\ (\text{trivial limit}),\\ H_{3,0}(w)*\overline{\Gamma}_r\left(h\left(-\frac{w}{K}\right)\right), & \text{if } K>0,\end{cases}$$

$$P(M_n^*\leq m)\overset{w}{\underset{n}{\rightarrow}}\begin{cases}H_{3,0}(m), & \text{if } K=0\ (\text{trivial limit}),\\ H_{3,0}(m)*\Gamma_r\left(h\left(\frac{m}{K}\right)\right), & \text{if } K>0,\end{cases}$$

where "$*$" denotes the convolution operator and the normalizing constants can be chosen such as $2A_{n:m}=A_{n:w}=a_{s_n}=\Psi_F(s_n+\sqrt{s_n})-\Psi_F(s_n)$ and $B_{n:w}=B_{n:m}=b_{s_n}=\Psi_F(s_n)$.

4. *If $F\in\mathcal{D}_{\mathcal{R}}(H_{3,0})$, $x^0=\infty$ and $\frac{\Psi_F(s_n+\sqrt{s_n})}{\Psi_F(s_n)}\underset{n}{\rightarrow}1$, then*

$$P(Q_n^*\leq q)\overset{w}{\underset{n}{\rightarrow}}P\left(\frac{1}{R_r}\leq q+1\right)\ (\text{trivial limit law}),$$

and

$$P(P_n^*\leq p)\overset{w}{\underset{n}{\rightarrow}}P(R_r\leq p+1)=\Gamma_r(h(p+1))\ (\text{trivial limit law}),$$

where $A_{n:q}=A_{n:p}=b_{s_n}=\Psi_F(s_n)$ and $B_{n:q}=B_{n:p}=b_{s_n}$.

Proof. First, by applying Theorem 29.2 of Billingsley (1979), which is particular case of a general result known as the continuous mapping theorem, we get the following basic limit relation

$$P(g(U_n,V_n)\le x)\xrightarrow[n]{w}P(g(U,V)\le x), \tag{3.1}$$

where $g(u,v)=u\pm v$, or $\frac{u}{v}$, or uv, $P(U_n\le x)\xrightarrow[n]{w}P(U\le x)$, $P(V_n\le y)\xrightarrow[n]{w}P(V\le y)$ and $P(U_n\le x)P(V_n\le y)\xrightarrow[n]{w}P(U\le x, V\le y)$. On the other hand, it is easy to check the validity of the following equalities:

$$W_n^*=\begin{cases}R_{s_n}^*-\dfrac{R_r}{a_{s_n}}, & \text{if } A_{n:w}=a_{s_n},\ B_{n:w}=b_{s_n}=0,\\ a_{s_n}b_{s_n}^{-1}R_{s_n}^*-\dfrac{R_r}{b_{s_n}}, & \text{if } A_{n:w}=b_{s_n},\ B_{n:w}=b_{s_n},\end{cases} \tag{3.2}$$

$$M_n^*=\begin{cases}R_{s_n}^*+\dfrac{R_r}{a_{s_n}}, & \text{if } 2A_{n:m}=a_{s_n},\ B_{n:m}=b_{s_n}=0,\\ a_{s_n}b_{s_n}^{-1}R_{s_n}^*+\dfrac{R_r}{b_{s_n}}, & \text{if } 2A_{n:m}=b_{s_n},\ B_{n:m}=b_{s_n},\end{cases} \tag{3.3}$$

$$Q_n^*=\begin{cases}\dfrac{R_{s_n}^*}{R_r}, & \text{if } A_{n:q}=a_{s_n},\ B_{n:q}=b_{s_n}=0,\\ \dfrac{a_{s_n}b_{s_n}^{-1}R_{s_n}^*-(R_r-1)}{R_r}, & \text{if } A_{n:q}=b_{s_n},\ B_{n:q}=b_{s_n},\end{cases} \tag{3.4}$$

$$P_n^*=\begin{cases}R_{s_n}^*R_r, & \text{if } A_{n:p}=a_{s_n},\ B_{n:p}=b_{s_n}=0,\\ a_{s_n}b_{s_n}^{-1}R_{s_n}^*R_r+(R_r-1), & \text{if } A_{n:p}=b_{s_n},\ B_{n:p}=b_{s_n}.\end{cases} \tag{3.5}$$

Now, by using (3.1), Theorems 1.1, 2.1 and Lemma 2.2.1 in Galambos (1987) (note that $a_{s_n}\xrightarrow[n]{}x^0=\infty$), the four limit relations in the first part of the theorem follow immediately from the first parts of (3.2)–(3.5), respectively. Also, the four limit relations in the second part of theorem follow from the second relations of (3.2)–(3.5), respectively, and Theorems 1.1, 2.1 (note that Theorem 1.1 implies $a_{s_n}b_{s_n}^{-1}\xrightarrow[n]{}0$, in Parts(a) and (b)). Moreover, the two limit relations in Part (3) follow from the first part of (3.2) and (3.3), respectively, and Theorem 1.1, where the condition $a_{s_n}^{-1}=(\Psi_F(s_n+\sqrt{s_n})-\Psi_F(s_n))^{-1}\xrightarrow[n]{}0$ implies the trivial limit (where $R_{s_n}^*$ outweighs $\frac{R_r}{a_{s_n}}$), while the condition $a_{s_n}^{-1}=(\Psi_F(s_n+\sqrt{s_n})-\Psi_F(s_n))^{-1}\xrightarrow[n]{}K>0$ implies the given non-trivial limit law. Finally, the two limit relations of Part (4) follow from the two equalities in the second part of (3.4) and (3.5), respectively, and Theorems 1.1, 2.1, where the condition $\frac{\Psi_F(s_n+\sqrt{s_n})}{\Psi_F(s_n)}\xrightarrow[n]{}1$ implies $a_{s_n}b_{s_n}^{-1}\xrightarrow[n]{}0$. □

Theorem 3.2. *Let $r_n\xrightarrow[n]{}\infty$.*

1. *If $F\in\mathcal{D}_{\mathcal{R}}(H_{1,\gamma})$ and $\frac{\Psi_F(r_n)}{\Psi_F(s_n)}\xrightarrow[n]{}c$, $1\ge c\ge 0$, then $P(W_n^*\le w)\xrightarrow[n]{w}H_{1,\gamma}(w)*\overline{H}_{1,\gamma}(-\frac{w}{c})$ and $P(M_n^*\le m)\xrightarrow[n]{w}H_{1,\gamma}(m)*H_{1,\gamma}(\frac{m}{c})$, where the trivial limit case occurs if $c=0$, since in this case R_{s_n} outweighs R_{r_n}. In this case, the normalizing constants can be taken as $2A_{n:m}=A_{n:w}=a_{s_n}=\Psi_F(s_n)$ and $B_{n:m}=B_{n:w}=0$. On the other hand, if $F\in\mathcal{D}_{\mathcal{R}}(H_{1,\gamma})$, then*

$$P(Q_n^*\le q)\xrightarrow[n]{w}\begin{cases}\displaystyle\int_0^\infty H_{1,\gamma}(qt)dH_{1,\gamma}(t), & q\ge 0,\\ 0, & q<0\end{cases}$$

and

$$P(P_n^*\le p)\xrightarrow[n]{w}\begin{cases}\displaystyle\int_0^\infty H_{1,\gamma}\left(\frac{p}{t}\right)dH_{1,\gamma}(t), & p\ge 0,\\ 0, & p<0,\end{cases}$$

where we can take $A_{n:q}=A_{n:p}=\frac{a_{s_n}}{a_{r_n}}=\frac{\Psi_F(s_n)}{\Psi_F(r_n)}$ and $B_{n:q}=B_{n:p}=0$.

2. *If $F\in\mathcal{D}_{\mathcal{R}}(H_{2,\gamma})$ and $d_{2:n}=\frac{a_{s_n}}{a_{r_n}}(=\frac{x^0-\Psi_F(s_n)}{x^0-\Psi_F(r_n)})\xrightarrow[n]{}d_2$, $1\ge d_2\ge 0$, then $P(W_n^*\le w)\xrightarrow[n]{w}H_{2,\gamma}(\frac{w}{d_2})*\overline{H}_{2,\gamma}(-w)$ and $P(M_n^*\le m)\xrightarrow[n]{w}H_{2,\gamma}(\frac{m}{d_2})*H_{2,\gamma}(m)$, where the trivial limit case occurs if $d_2=0$ (since, R_{r_n} outweighs R_{s_n}). In this case, we can take $2A_{n:m}=A_{n:w}=a_{r_n}=x^0-\Psi_F(r_n)$ and $B_{n:m}=B_{n:w}=b_{s_n}-b_{r_n}=x^0-x^0=0$. Moreover, if $x^0\ne 0$,*

$$P(Q_n^*\le q)\xrightarrow[n]{w}\begin{cases}H_{2,\gamma}(q)*\overline{H}_{2,\gamma}\left(-\dfrac{q}{d_2}\right), & x^0>0,\\ H_{2,\gamma}\left(\dfrac{q}{d_2}\right)*\overline{H}_{2,\gamma}(-q), & x^0<0.\end{cases}$$

In this case, the normalizing constants can be taken as $A_{n:q}=\frac{a_{r_n}}{|b_{s_n}|}=\frac{a_{r_n}}{|x^0|}$ *and* $B_{n:q}=\frac{b_{r_n}}{b_{s_n}}=1$*. Finally, if* $x^0\neq 0$,

$$P(P_n^*\le p)\underset{n}{\overset{w}{\to}}\begin{cases} H_{2,\gamma}\left(\dfrac{p}{d_2}\right)*H_{2,\gamma}(p), & x^0>0,\\ \overline{H}_{2,\gamma}\left(-\dfrac{p}{d_2}\right)*\overline{H}_{2,\gamma}(-p), & x^0<0,\end{cases}$$

where the normalizing constants can be chosen such as $A_{n:p}=a_{r_n}|b_{s_n}|=a_{r_n}|x^0|$ *and* $B_{n:p}=b_{r_n}b_{s_n}=(x^0)^2$.

3. *If* $F\in\mathcal{D}_{\mathcal{R}}(H_{3,0})$ *and* $d_{3:n}=\frac{a_{s_n}}{a_{r_n}}(=\frac{\Psi_F(s_n+\sqrt{s_n})-\Psi_F(s_n)}{\Psi_F(r_n+\sqrt{r_n})-\Psi_F(r_n)})\underset{n}{\to}d_3,\ 0\le d_3<\infty$ *(for the case* $d_3=\infty$*, see Remark 3.1), then* $P(W_n^*\le w)\underset{n}{\overset{w}{\to}}H_{3,0}(\frac{w}{d_3})*\overline{H}_{3,0}(-w)$ *and* $P(M_n^*\le m)\underset{n}{\overset{w}{\to}}H_{3,0}(\frac{m}{d_3})*H_{3,0}(m)$*, where the trivial limit case occurs if* $d_3=0$*, since in this case* R_{r_n} *outweighs* R_{s_n}*. In this case, we can take* $2A_{n:m}=A_{n:w}=a_{r_n}=\Psi_F(r_n+\sqrt{r_n})-\Psi_F(r_n)$ *and* $B_{n:m}=B_{n:w}=b_{s_n}-b_{r_n}$*. Moreover, if* $F\in\mathcal{D}_{\mathcal{R}}(H_{3,0})$*,* $\frac{\Psi_F(s_n+\sqrt{s_n})}{\Psi_F(s_n)}\underset{n}{\to}1$ *and* $\ell_n=\frac{a_{s_n}b_{r_n}}{a_{r_n}b_{s_n}}(=\frac{(\Psi_F(s_n+\sqrt{s_n})-\Psi_F(s_n))\Psi_F(r_n)}{(\Psi_F(r_n+\sqrt{r_n})-\Psi_F(r_n))\Psi_F(s_n)})\underset{n}{\to}\ell,\ 0\le\ell<\infty$ *(for the case* $\ell=\infty$*, see Remark 3.1), then*

$$P(Q_n^*\le q)\underset{n}{\overset{w}{\to}}\begin{cases} H_{3,0}(q)*\overline{H}_{3,0}\left(-\dfrac{q}{\ell}\right), & x^0>0,\\ H_{3,0}\left(\dfrac{q}{\ell}\right)*\overline{H}_{3,0}(-q), & x^0<0.\end{cases}$$

In this case, the normalizing constants can be chosen such as $A_{n:q}=\frac{a_{r_n}}{|b_{s_n}|}$ *and* $B_{n:q}=\frac{b_{r_n}}{b_{s_n}}$*. Finally, If* $F\in\mathcal{D}_{\mathcal{R}}(H_{3,0})$ *and* $\ell_n\underset{n}{\to}\ell,\ 0\le\ell<\infty$,

$$P(P_n^*\le p)\underset{n}{\overset{w}{\to}}\begin{cases} H_{3,0}\left(\dfrac{p}{\ell}\right)*H_{3,0}(p), & x^0>0,\\ \overline{H}_{3,0}\left(-\dfrac{p}{\ell}\right)*\overline{H}_{3,0}(-p), & x^0<0,\end{cases}$$

where the normalizing constants can be chosen such as $A_{n:p}=a_{r_n}|b_{s_n}|$ *and* $B_{n:p}=b_{r_n}b_{s_n}$.

Proof. First, it is easy to check the validity of the following equalities:

$$W_n^*=\begin{cases} R_{s_n}^*-\dfrac{a_{r_n}}{a_{s_n}}R_{r_n}^*, & \text{if } A_{n:w}=a_{s_n},\ B_{n:w}=b_{s_n}=0,\\ \dfrac{a_{s_n}}{a_{r_n}}R_{s_n}^*-R_{r_n}^*, & \text{if } A_{n:w}=a_{r_n},\ B_{n:w}=b_{s_n}-b_{r_n},\end{cases}\tag{3.6}$$

$$M_n^*=\begin{cases} R_{s_n}^*+\dfrac{a_{r_n}}{a_{s_n}}R_{r_n}^*, & \text{if } 2A_{n:m}=a_{s_n},\ B_{n:m}=b_{s_n}=0,\\ \dfrac{a_{s_n}}{a_{r_n}}R_{s_n}^*+R_{r_n}^*, & \text{if } 2A_{n:m}=a_{r_n},\ B_{n:m}=b_{s_n}-b_{r_n},\end{cases}\tag{3.7}$$

$$Q_n^*=\begin{cases} \dfrac{R_{s_n}^*}{R_{r_n}^*}, & \text{if } A_{n:q}=\dfrac{a_{s_n}}{a_{r_n}},\ B_{n:q}=b_{s_n}=0,\\ -\dfrac{\ell_nR_{s_n}^*-R_{r_n}^*}{|b_{s_n}|^{-1}R_{s_n}}, & \text{if } A_{n:q}=\dfrac{a_{r_n}}{|b_{s_n}|},\ B_{n:q}=\dfrac{b_{r_n}}{b_{s_n}},\end{cases}\tag{3.8}$$

$$P_n^*=\begin{cases} R_{s_n}^*R_{r_n}^*, & \text{if } A_{n:p}=\dfrac{a_{s_n}}{a_{r_n}},\ B_{n:p}=b_{s_n}=0,\\ \dfrac{a_{s_n}}{b_{s_n}}R_{s_n}^*R_{r_n}^*+\ell_nR_{s_n}^*+R_{r_n}^*, & \text{if } A_{n:p}=a_{r_n}b_{s_n},\ B_{n:p}=b_{r_n}b_{s_n},\ x^0>0,\\ \dfrac{a_{s_n}}{|b_{s_n}|}R_{s_n}^*R_{r_n}^*-\ell_nR_{s_n}^*-R_{r_n}^*, & \text{if } A_{n:p}=a_{r_n}|b_{s_n}|,\ B_{n:p}=b_{r_n}b_{s_n},\ x^0<0,\end{cases}\tag{3.9}$$

where the last two equalities in (3.9) are valid for large n (note that, since $b_{r_n},\ b_{s_n}\underset{n}{\to}x^0$, then for large n, $\text{sign}(b_{r_n}),\ \text{sign}(b_{s_n})=\text{sign}(x^0)$). Now, by using (3.1), Theorems 1.1, 2.1 and the condition $\frac{\Psi_F(r_n)}{\Psi_F(s_n)}\underset{n}{\to}c,\ 1\ge c\ge 0$ (note that, since the function $\Psi_F(n)$ is non-decreasing, then for large n, we get $a_{r_n}<a_{s_n}$ and $0\le\frac{a_{r_n}}{a_{s_n}}=\frac{\Psi_F(r_n)}{\Psi_F(s_n)}\le 1$), the four limit relations in the first part of the theorem follow immediately from the first part of the relations (3.6)–(3.9), respectively. On the other hand, for the statistic Q_n^* in the second and third parts of theorem we have $\frac{b_{s_n}}{a_{s_n}}\underset{n}{\to}\infty$ (this limit relation is satisfied in the third part due

to the condition $\frac{\Psi_F(s_n+\sqrt{s_n})}{\Psi_F(s_n)} \underset{n}{\longrightarrow} 1$). Therefore, by applying Lemma 3.3 in Barakat (1998), we get

$$\frac{R_{s_n}}{|b_{s_n}|} \overset{p}{\underset{n}{\to}} \begin{cases} +1, & \text{if } x^0 > 0, \\ -1, & \text{if } x^0 < 0, \end{cases} \tag{3.10}$$

where $\overset{p}{\underset{n}{\to}}$ means convergence in probability, as $n \to \infty$. Now, by using (3.1), (3.10) (for the statistic Q_n^*), Theorems 1.1, 2.1, the relation $\frac{a_{s_n}}{b_{s_n}} \underset{n}{\longrightarrow} 0$, and the condition $\ell_n = d_{2:n} = \frac{x^0-\Psi_F(s_n)}{x^0-\Psi_F(r_n)} \underset{n}{\longrightarrow} d_2$, $1 \geq d_2 \geq 0$ (note that, since the function $\Psi_F(n)$ is non-decreasing, then for large n, we get $a_{s_n} < a_{r_n}$ and $0 \leq d_{2:n} = \frac{a_{s_n}}{a_{r_n}} \leq 1$), the four limit relations in the second part of the theorem follow immediately from the second part of the relations (3.6)–(3.9), respectively. Finally, by using (3.1), (3.10) (for the statistic Q_n^*), Theorems 1.1, 2.1, the relation $\frac{a_{s_n}}{b_{s_n}} \underset{n}{\longrightarrow} 0$ and the condition $\ell_n \underset{n}{\longrightarrow} \ell$, the four limit relations in the third part of the theorem follow immediately from the second part of the relations (3.6)–(3.9), respectively. □

Remark 3.1. If $d_3 = \infty$, we get the trivial convergence $P(W_n^* \leq x), P(M_n^* \leq x) \overset{w}{\underset{n}{\to}} H_{3,0}(x)$, since in this case R_{s_n} outweighs R_{r_n}. This fact, can easily be verified if we take the normalizing constants $2A_{n:m} = A_{n:w} = a_{s_n} = \Psi_F(s_n + \sqrt{s_n}) - \Psi_F(s_n)$ and $B_{n:m} = B_{n:w} = b_{s_n} - b_{r_n}$, to get the equalities $W_n^* = R_{s_n}^* - \frac{a_{r_n}}{a_{s_n}} R_{r_n}^*$ and $M_n^* = R_{s_n}^* + \frac{a_{r_n}}{a_{s_n}} R_{r_n}^*$. On the other hand, clearly, if $\ell_n \underset{n}{\longrightarrow} \ell = 0$, we get the trivial convergence

$$P(Q_n^* \leq q) \overset{w}{\underset{n}{\to}} \begin{cases} H_{3,0}(q), & x^0 > 0, \\ \overline{H}_{3,0}(-q), & x^0 < 0, \end{cases} \tag{3.11}$$

since in this case R_{r_n} outweighs R_{s_n}. However, if $\ell = \infty$, we get also the same trivial convergence (3.11), but in this case R_{s_n} outweighs R_{r_n}. This result can easily be seen, if we use the normalizing constants $A_{n:q} = \frac{a_{s_n}}{|b_{r_n}|}$ and $B_{n:q} = \frac{b_{s_n}}{b_{r_n}}$, in order to get the equality $Q_n^* = \frac{R_{s_n}^* - \ell_n^{-1} R_{r_n}^*}{|b_{r_n}|^{-1} R_{r_n}}$.

Corollary 3.1. *By virtue of Theorem 3.2, we can deduce an important fact that in most cases of the convergence (specially, for the 2nd and the 3rd parts), we have the asymptotic relations $Q_n^* \overset{w}{\underset{n}{=}} \pm W_n^*$ and $P_n^* \overset{w}{\underset{n}{=}} \pm M_n^*$, where $X_n \overset{w}{\underset{n}{=}} Y_n$ means that both the df's of random sequences X_n and Y_n weakly converge to the same limit. This fact has considerable practical importance.*

Example 3.1. For the Weibull distribution, $F(x) = P(X \leq x) = 1 - e^{-x^\alpha}$, $x, \alpha > 0$, we can easily show that $\Psi_F(u) = u^{\frac{1}{\alpha}}$. Therefore, $\frac{\Psi_F(n+\sqrt{n})}{\Psi_F(n)} = (1 + \frac{1}{\sqrt{n}})^{\frac{1}{\alpha}} \underset{n}{\to} 1$. Moreover, $\Psi_F(n + \sqrt{n}) - \Psi_F(n) = (n + \sqrt{n})^{\frac{1}{\alpha}} - n^{\frac{1}{\alpha}} = n^{\frac{1}{\alpha}} \frac{1}{\alpha\sqrt{n}}(1 + \circ(1))$. Thus $\Psi_F(n + \sqrt{n}) - \Psi_F(n) \underset{n}{\to} \frac{1}{\alpha}$, if $\alpha = 2$ and $\Psi_F(n + \sqrt{n}) - \Psi_F(n) \underset{n}{\to} \infty$, if $\alpha > 2$. Thus, if $r_n = r =$ constant, Theorem 3.1 implies

$$P(W_n^* \leq w) \overset{w}{\underset{n}{\to}} \begin{cases} H_{3,0}(w), & \text{if } \alpha > 2, \\ H_{3,0}(w) * \left(\overline{\Gamma}_r\left(\frac{w^2}{4}\right) \mathrm{I}_{(-\infty,0)}(w)\right), & \text{if } \alpha = 2, \end{cases}$$

$$P(M_n^* \leq m) \overset{w}{\underset{n}{\to}} \begin{cases} H_{3,0}(m), & \text{if } \alpha > 2, \\ H_{3,0}(m) * \left(\Gamma_r\left(\frac{m^2}{4}\right) \mathrm{I}_{(0,\infty)}(m)\right), & \text{if } \alpha = 2, \end{cases}$$

$P(Q_n^* \leq q) \overset{w}{\underset{n}{\to}} P(\frac{1}{R_r} \leq q + 1)$, and $P(P_n^* \leq p) \overset{w}{\underset{n}{\to}} \Gamma_r((p + 1)^2)$, where $\mathrm{I}_A(x)$ is the usual indicator function. On the other hand, if $\frac{r_n}{s_n} \underset{n}{\to} \ell^2$, $0 < \ell \leq 1$ and $s_n - r_n \underset{n}{\to} \infty$ (clearly, the relation $\frac{r_n}{s_n} \underset{n}{\to} \ell^2$, $0 < \ell < 1$, implies $s_n - r_n \underset{n}{\to} \infty$), we get, $d_{3:n} \underset{n}{\to} \ell^{\frac{\alpha-2}{\alpha}}$ and $\ell_n \underset{n}{\to} \ell^2$. Therefore, Theorem 3.2, implies that $P(W_n^* \leq w) \overset{w}{\underset{n}{\to}} H_{3,0}(\ell^{\frac{2-\alpha}{\alpha}} w) * \overline{H}_{3,0}(-w)$, $P(M_n^* \leq m) \overset{w}{\underset{n}{\to}} H_{3,0}(\ell^{\frac{2-\alpha}{\alpha}} m) * H_{3,0}(m)$, $P(Q_n^* \leq q) \overset{w}{\underset{n}{\to}} H_{3,0}(q) * \overline{H}_{3,0}(-\frac{q}{\ell^2})$and $P(P_n^* \leq p) \overset{w}{\underset{n}{\to}} H_{3,0}(p) * H_{3,0}(\frac{p}{\ell^2})$.

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